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Non-universality of random walks in random environments

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Abstract. Random walks with random transition probabilities are studied for walks on the Euclidean and fractal lattices. The randomness is defined by selecting a preferred local direction and then by establishing a probability to violate this local instruction. Monte Carlo and decimation methods are used. Our results suggest that $d = 2$ is the upper critical dimensionality for this problem. For $1 < d < 2$ the transport is subdiffusive and non-universal. For $d < 2$ no simple relation between the mean-square displacement and the probability of returning to the origin is found, in contrast to the ordinary random walk case.

Recently, random walks with random transition probabilities, the so-called random walks in random environments or random-random walks, have been studied by, e.g., Alexander *et al* (1981), Derrida and Pomeau (1982), Luck (1983), Derrida and Luck (1983), Fisher (1984), Fisher *et al* (1985) and Honkonen *et al* (1987). The random-random walks are believed to be related to transport phenomena in disordered media and to vibrations of lattices with random force constants (see the review article of Alexander *et al* (1981)). There are some exact results, the existing ones concerning mainly the one-dimensional case. In this case, asymptotic formulae describing the mean-square displacement and other important characteristics are known for a wide class of transition probability distributions (Alexander *et al* 1981, Derrida and Pomeau 1982). Sinai (1982) proved a general theorem that if a probability $p(x)$ to move to the right from any point x fulfils the condition

$$\langle \ln(p(x)) \rangle = \langle \ln(1 - p(x)) \rangle \tag{1}$$

then, after time t, a particle will almost certainly cover a distance $R \sim \ln^2 t$. This means that

$$\langle R^2(t) \rangle \sim \ln^4 t. \tag{2}$$

For higher lattice dimensionalities, results are scant. Within the renormalisation group method approach it has been argued that $d = 1$ and $d = 2$ are respectively the upper and lower critical dimensionalities in the random-random walk problem (Luck 1983, Derrida and Luck 1983, Fisher 1984, Fisher *et al* 1985). It means that, in the random-random walk case, classical scaling laws for the mean-square displacement are not modified for $d > 2$, there are logarithmic corrections for $d = 2$ and non-classical exponents for $1 < d < 2$, while peculiar phenomena may occur at $d = 1$. Fisher (1984), Fisher *et al* (1985) and Honkonen *et al* (1987) also discussed possible forms of asymptotic formulae of $\langle R^2(t) \rangle$ for $d = 2$.

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Numerical simulations of Marinari *et al* (1983a, b) and Banavar and Willemsen (1983) corroborated Sinai's results in the $d = 1$ case. Marinari *et al* (1983b) investigated also the two-dimensional case. They studied models in which on each site the probability $P_i(k)$ to move to its ith nearest neighbour is given by

$$P_i(k) = Q_i^k / Z(k) \tag{3}$$

$$Z(k) = \sum Q_i^k \tag{4}$$

where Q_i are the random numbers uniformly distributed in $(0, 1)$. Within the error bars their results for $d = 2$ are consistent with

$$\langle R^2(t) \rangle \sim t^{\alpha} \tag{5}$$

where the value of α depends on the form of the probability distribution function for the transition rates (i.e. on the value of k).

In the case of ordinary random walks the probability to return to the origin and the mean-square displacement are related and

$$P_0(t) \sim \langle R^2(t) \rangle^{-d/2} \tag{6}$$

where d denotes lattice dimensionality which, in the case of a fractal lattice, corresponds to its fractal dimensionality d_f (Alexander and Orbach 1982, Rammal and Toulouse 1982, Guyer 1984a, b).

In this paper we discuss results of numerical simulations of random-random walks for $d = 2, 3$ Euclidean lattices and for the Sierpinski gasket of fractal dimensionality $d_f = 1.5646\ldots$ (see figure 1). We find that in the case of random-random walks (6) does not seem to hold and the exponent α in (5) appears to be parameter dependent.

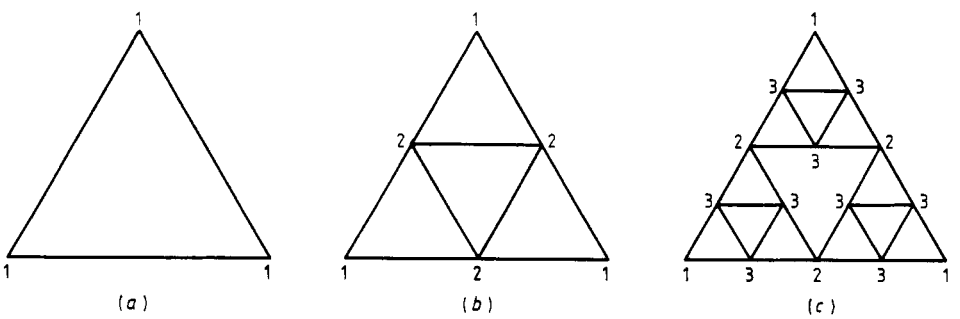

Figure 1. The first three steps of the Sierpinski gasket construction are shown.

A numerical realisation of a random-random walk may be defined in the following way. For each site of a lattice a direction pointing to one of its nearest neighbours (an 'instruction') is chosen randomly. A particle moves in this direction with probability $1 - p$ or in any other direction with probability $p/(z-1)$, where z is the coordination number. Thus by changing p, the probability to violate the instruction, it is possible to pass from completely deterministic ($p = 0$) to ordinary random walks ($p = p_0 = (z-1)/z$). For $p < p_0$ the 'instruction' specifies a favourite direction of motion. In a numerical simulation, results have to be averaged over different walks and different instruction sets corresponding to a preassigned p. In a $d = 1$ case our model coincides with that of Marinari *et al* (1983a, b) but differs for $d > 1$. Results of our calculations

for Euclidean lattices of dimensionality two and three are shown in figures 2(*a*) and 2(*b*) respectively. In the $d=3$ case the asymptotic region of $\langle R^2(t)\rangle \sim t$ is reached for each value of $p \neq 0$, while in the $d=2$ case some curvature persists in the plot of $\ln(\langle R^2(t)\rangle)$ against t. This indicates that in the asymptotic region there exist non-algebraic corrections to the $\langle R^2(t)\rangle \sim t$ type of behaviour, suggesting $d=2$ to be the upper critical dimensionality. In the $d=2$ case we observed no asymptotic relation of the type

$$\langle R^2(t)\rangle \sim t(1+4/\ln(t)) \tag{7}$$

predicted by Fisher (1984) for his version of the random-random walk. To further investigate this point we also made Monte Carlo simulations of random walks with step lengths randomly chosen from a Gaussian distribution. This model seemed to correspond better to the case considered by Fisher (1984) than the previous one. None of the formulae discussed by Fisher (1984) and Fisher *et al* (1985) fits our data. Note, however, that these results were obtained for the weak disorder limit and this condition is rather difficult to fulfil in our numerical model.

Consider now the case of $d<2$. Figure 3 shows $\langle R^2(t)\rangle$ for the case of the Sierpinski gasket. The results are consistent with the power law asymptotic formula

$$\langle R^2(t)\rangle \sim t^{\alpha}. \tag{8}$$

The exponent α appears to be p dependent, at least, within the 2^{16} steps considered. When $p=p_0$, this exponent becomes equal to $2d_w=2d_f/d_s$ (d_s denotes the spectral

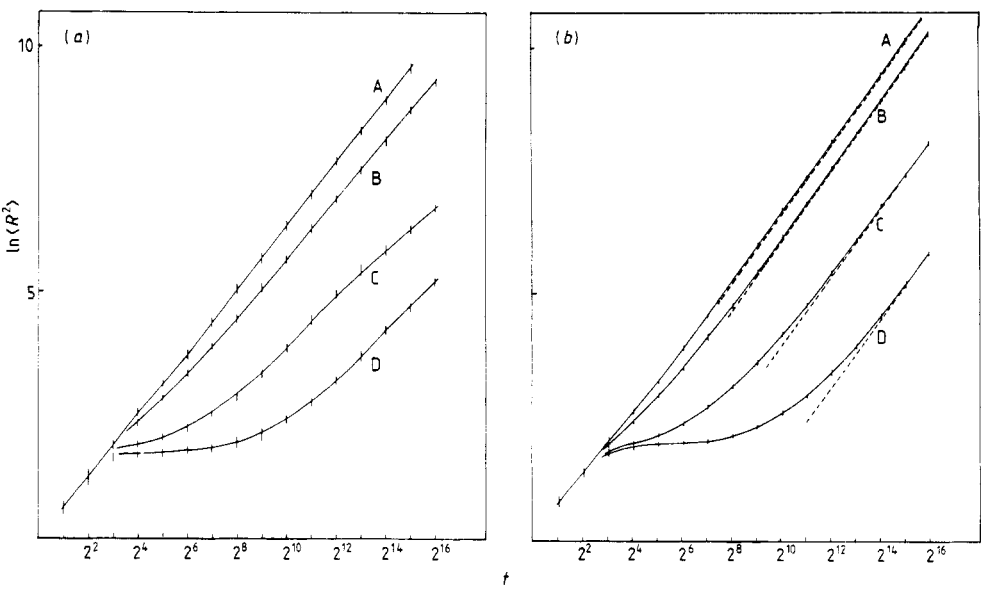

Figure 2. The mean-square displacement as a function of t (time = number of steps) for random-random walks on Euclidean lattices: A, $p=0.2$; B, $p=0.1$; C, $p=0.01$; D, $p=0.001$. (*a*) $d=2$ case. Each point corresponds to an average of five runs of the Monte Carlo program. In each run 30 sets of instructions on a 500×500 lattice and 100 walks for every set were generated. (*b*) $d=3$ case. Each point is an average of three runs of 3000 walks (30 instruction sets, 100 walks for each set). Instructions are generated on a $65\times65\times65$ lattice (periodic boundary conditions are assumed). The broken lines correspond to $\langle R^2(t)\rangle \sim t$ type asymptotic behaviour.

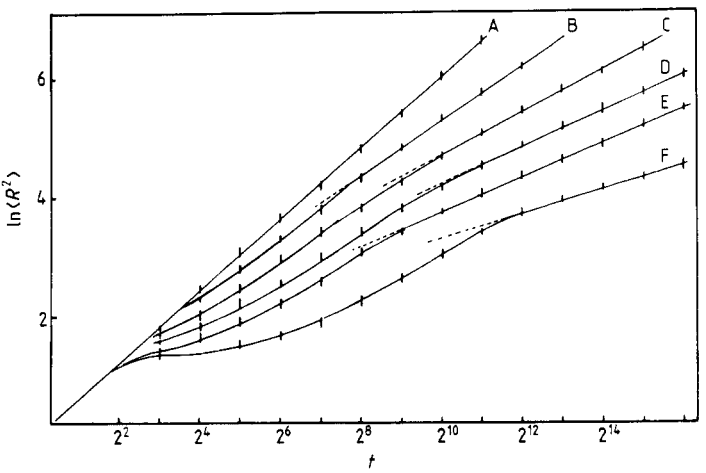

Figure 3. The mean-square displacement as a function of t for the Sierpinski gasket. Each point is an average of five runs of 1000 walks. A, $p = 0.75$, $\alpha = 0.86$; B, $p = 0.25$, $\alpha = 0.64$; C, $p = 0.125$, $\alpha = 0.52$; D, $p = 0.0625$, $\alpha = 0.44$; E, $p = 0.0315$, $\alpha = 0.42$; F, $p = 0.008$, $\alpha = 0.26$. . . . , $\langle R^2\rangle = t^{\alpha}$.

dimensionality) which corresponds to the standard random walk on this fractal lattice. For other values of p, α is less than the standard value.

We now focus on the probability of returning to the origin. This characteristic is difficult to study by a direct Monte Carlo simulation technique since this would require an unreasonably large amount of computer time to accumulate reasonable statistics. However, for walks on a chain and on the Sierpinski gasket it is possible to use a length scale renormalisation method applied by Guyer (1984a, b) to study ordinary random walks on fractals. In the short description of this method given below we concentrate only on a $d = 1$ case—the generalisation needed for the Sierpinski gasket is straightforward.

Probabilities $P(n, t)$ for a particle to occupy site n at time t are described by

$$\frac{\mathrm{d}P(n,t)}{\mathrm{d}t} = W_{n+1,n}P(n+1,t) + W_{n-1,n}P(n-1,t) - V_nP(n,t) \tag{9}$$

where $W_{n\pm1,n} = p$ or $1-p$ and $V_n = 1$. If we apply the Laplace transformation to (9) assuming

$$P(n,0) = \delta_{n0} \tag{10}$$

as an initial condition, we get

$$s\mathscr{P}(n,s) = \delta_{n0} - V_n\mathscr{P}(n,s) + W_{n+1,n}\mathscr{P}(n+1,s) + W_{n-1,n}\mathscr{P}(n-1,s) \tag{11}$$

where $\mathscr{P}(n,s) = \int_0^{\infty} \mathrm{e}^{-st}P(n,t)\,\mathrm{d}t$. Equations for $\mathscr{P}(n,s)$ with n odd are then substituted into equations with even n. Thus a new set of equations (connecting only the occupation probabilities for even n) with new effective transition rates

$$V_n^{(1)} = (W_{n+1,n}W_{n,n+1} + W_{n-1,n}W_{n,n-1})/(s+V_n) \tag{12}$$

$$W_{n+2,n}^{(1)} = W_{n+2,n+1}W_{n+1,n}/(s+V_n) \tag{13}$$

$$W_{n-2,n}^{(1)} = W_{n-2,n-1}W_{n-1,n}/(s+V_n) \tag{14}$$

is obtained. Repetition of this procedure leads in the kth step to a set of equations for $\mathscr{P}(n, s)$ with $n = \pm 2^k l$ where $l = 0, 1, 2, 3, \ldots$. In the case of the Sierpinski gasket each step consists of elimination of the innermost sites (sites '3' in figure 1 for the gasket of this size). This again is equivalent to doubling of the length scale and to renormalising the transition rates. Since, with increasing number of steps, effective transition rates $W^{(k)}$ converge to zero while the average of $V^{(k)}$ converges to a constant $V^{(\infty)}$ we have

$$\mathscr{P}(0, s) = 1/(s + V^{(\infty)}). \tag{15}$$

Thus for each p and any set of instructions after a few steps only we get $V^{(\infty)}$ and hence $\mathscr{P}(0, s)$.

Figures 4(a) and (b) display $V^{(\infty)}$ (averaged over different sets of instructions) as a function of s for the $d = 1$ system and for the Sierpinski gasket respectively. For ordinary random walks $P_0(t)$ and the mean-square displacement are related according to equation (6). If the same identity held for random-random walks then for large t we would have

$$P(0, t) = P_0(t) \sim \langle R^2(t) \rangle^{-1/2} \sim (\ln^2 t)^{-1} \tag{16}$$

in the 1D case and

$$P(0, t) = P_0(t) \sim t^{-\alpha(p)d/2} \tag{17}$$

in the Sierpinski gasket case (assuming that $\langle R^2(t) \rangle \sim t^{\alpha}$ in this case). According to Tauberian theorems (see, e.g., the book by Doetsch (1950)) equations (16) and (17) are equivalent to

$$\mathscr{P}(0, s) \sim 1/(s \ln^2 s) \qquad \text{or} \qquad V^{(\infty)} \sim s \ln^2 s \tag{18}$$

for $d = 1$ and

$$\mathscr{P}(0, s) \sim s^{\alpha(p)d/2 - 1} \qquad \text{or} \qquad V^{(\infty)} \sim s^{1 - \alpha(p)d/2} \tag{19}$$

for $d = 1.5846\ldots$ in the limit of small s. Both types of asymptotic behaviour are inconsistent with the results obtained using the decimation method for $p \neq p_0$ (compare figures 4(a) and (b)) where the observed decrease of $V^{(\infty)}$ with decreasing s is much slower than described by (18) and (19). Note, for instance, that according to equation (19) $V^{(\infty)}$ should change with s faster in the $p = 0.008$ case than in the $p = 0.25$ one (since $\alpha(0.008) < \alpha(0.25)$)—cf figure 3) in contradiction to the results shown in figure 4(b). Although we cannot exclude that in our calculations the asymptotic region is not reached we find this possibility rather unlikely. Therefore we conclude that in the random-random walk case $P_0(t)$ and $\langle R^2(t) \rangle$ are not related according to equation (6). This may indicate that there exist independent scaling laws for $P_0(t)$ and $\langle R^2(t) \rangle$. On the other hand the evolution of $V^{(n)}$ and $W^{(n)}$ with the number of decimation steps are qualitatively similar in the 1D and the Sierpinski gasket case (examples of this evolution for random-random and ordinary random walks are shown in figures 5(a) and (b).

Our numerical simulations of the problem of random walks in random environments are consistent with the renormalisation group method results that for $d > 2$ there is the ordinary diffusive behaviour while for $d < 2$ a power law of the type $\langle R^2(t) \rangle \sim t^{\alpha}$ with non-universal value of α is observed. For the $d = 2$ case non-algebraic corrections to the $\langle R^2(t) \rangle$ against t dependence are found although their form seems to be more complicated than suggested by Fisher (1984) and Fisher *et al* (1985) for the weak

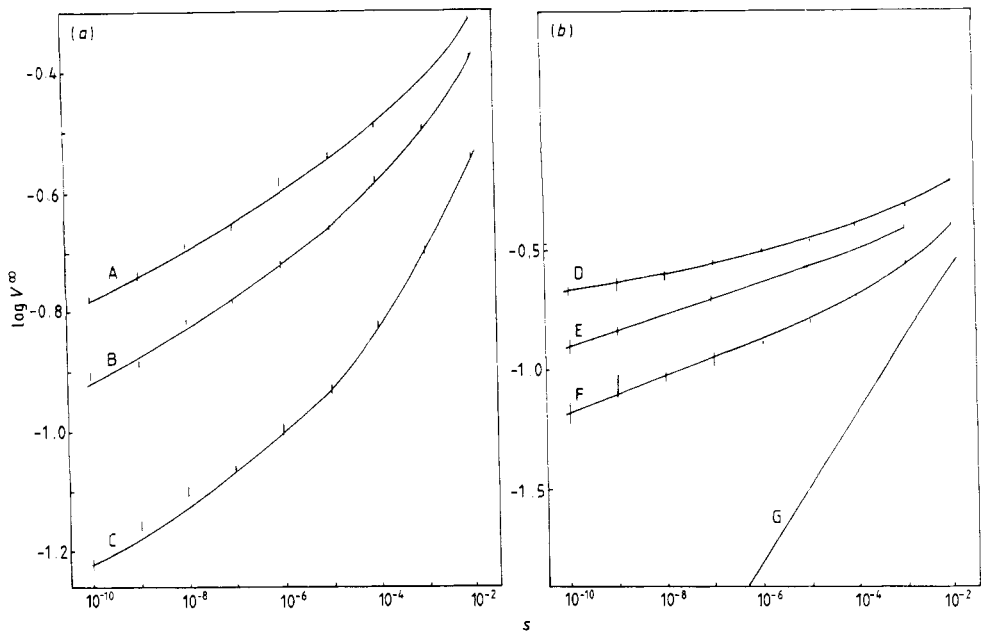

Figure 4. $V^{(\infty)}$ as a function of s for (a) a one-dimensional system; (b) the Sierpinski gasket. A, $p = 0.001$; B, $p = 0.01$; C, $p = 0.1$; D, $p = 0.008$; E, $p = 0.0625$; F, $p = 0.25$, G, $p = 0.75$.

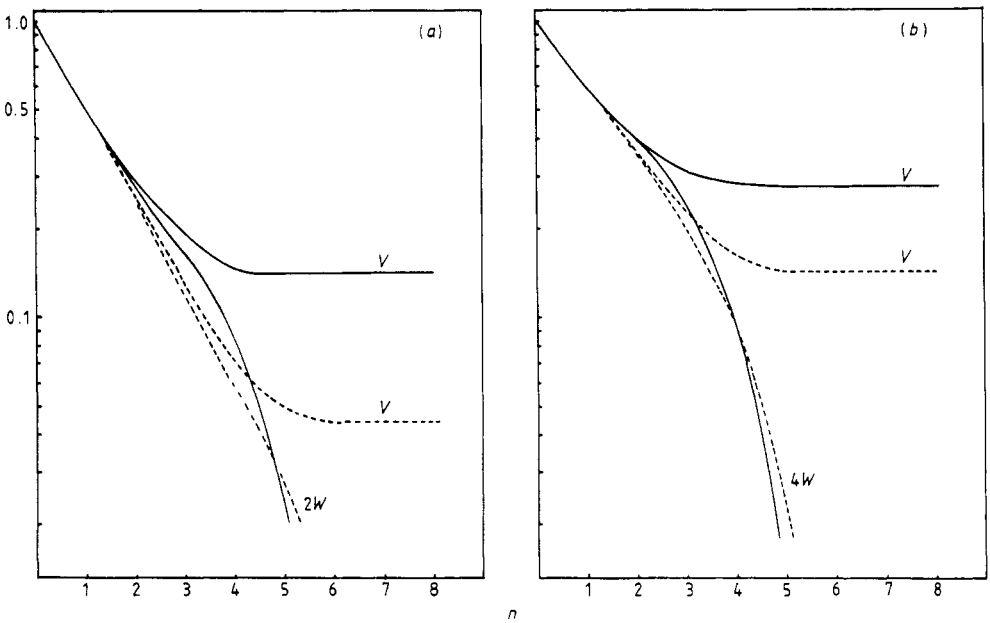

Figure 5. Evolution of $V^{(n)}$ and $W^{(n)}$ with the number of decimation steps n for (a) a one-dimensional system ——, $p = 0.2$, - - - $p = 0.5$; (b) the Sierpinski gasket —— $p = 0.25$, - - - $p = 0.75$. Each value of V and W (zW is displayed, where $z = 2$ or 4 is the coordination number) is averaged over all equivalent sites in a given decimation step. Random-random walk (full curve) and ordinary random walk cases (broken curve) are compared for $s = 0.001$.

disorder limit. Furthermore our results indicate that there is no simple relation between $P_0(t)$ and the mean-square displacement in the random-random walk case. The existence of independent scaling laws for different characteristics is related to the fact that walks of the type discussed are characterised by an infinite number of critical exponents (Bunde 1987).

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